PROBLEM SET III, PROBLEMS I, II

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Problem 1. Let $E \subseteq \mathbb{R}^n$ be a measurable set with $\mu(E) < \infty$. Show that for each $\epsilon > 0$, there exists a set $E' \subseteq \mathbb{R}^n$ which is a finite disjoint union of open boxes satisfying

$$\mu(E - E'), \mu(E' - E) < \epsilon.$$

Proof. From Problem Set II, we know that, for $\epsilon > 0$, there exists a compact set $K \subseteq E$ such that

$$\mu(E) - \epsilon \le \mu(K) \le \mu(E).$$

Now let us take a covering of K with countably many open boxes B_i such that

$$\mu(K) \le \mu\left(\bigcup_{i=0}^{\infty} B_i\right) \le \mu(K) + \epsilon.$$

By compactness, we reduce this to a finite subcover of K given by

$$E' = \bigcup_{k=0}^{n} B_k.$$

By the subdivision argument given in lecture, we assume these boxes to be disjoint. The measure of the closure of the finite subcover will be equivalent to the measure of the subcover itself, as the boundaries of the boxes have measure zero, and the closure contains K. Thus,

$$\mu(E) - \epsilon \le \mu(K) \le \mu(E') \le \mu(K) + \epsilon.$$

Finally,

$$\mu(E - E') = \mu(E) - \mu(E \cap E') \le \mu(E) - \mu(K) \le \epsilon,$$

and

$$\mu(E' - E) = \mu(E') - \mu(E' \cap E) \le \mu(E') - \mu(K) \le \epsilon,$$

which is precisely what we wished to prove.

□

Problem 2. Let $f_1, f_2, ... : \mathbb{R}^n \to \mathbb{R}$ be a sequence of measurable functions and suppose that for each $\vec{x} \in \mathbb{R}^n$, the sequence $\{f_i(\vec{x})\}$ is bounded. Show that the function $f(\vec{x}) = \limsup \{f_i(\vec{x})\}$ is measurable.

Proof. For each $\vec{x} \in \mathbb{R}^n$, we have $f(\vec{x}) \le \alpha$ if and only if the non-increasing sequence $\sup_{m \ge M} f_m(\vec{x})$ converges to some number $\le \alpha$ as $M \to \infty$. That is, $f(\vec{x}) \le \alpha$ if and only if, for each $n \in \mathbb{Z}_+$, we have

$$\sup_{m \ge M} f_m(\vec{x}) \le \alpha + \frac{1}{n}$$

for M sufficiently large. Equivalently, $f(\vec{x}) \le \alpha$ if and only if, for each $n \in \mathbb{Z}_+$, we find an M large enough such that

$$f_m(\vec{x}) \le \alpha + \frac{1}{n}$$

for all $m \ge M$. Hence,

$$\{\vec{x} \in \mathbb{R}^n : f(\vec{x}) \le \alpha\}$$

is equivalent to

$$\bigcap_{n=1}^{\infty} \bigcup_{M=1}^{\infty} \bigcap_{m>M} \left\{\vec{x} \in \mathbb{R}^n : f_m(\vec{x}) \le \alpha + \frac{1}{n}\right\},$$

which is measurable since each f_m is measurable and the countable intersection or union of measurable sets is also measurable. Therefore, f is measurable, and we are done. □